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BASIC RESEARCH IN THE MATHEMATICAL FOUNDATIONS OF STABILITY THE--ETC(U)
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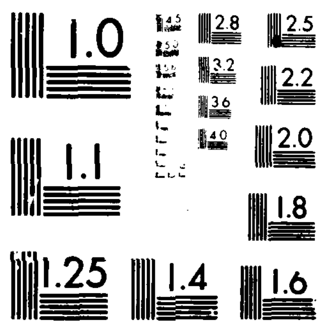

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ANNUAL SCIENTIFIC REPORT

on

Contract AF F4962078-C-0030

BASIC RESEARCH IN THE MATHEMATICAL FOUNDATIONS OF

STABILITY THEORY, CONTROL THEORY AND NUMERICAL

LINEAR ALGEBRA

October 1, 1978 - September 30, 1979

Principal Investigator: Marvin Marcus

Co-Investigators: Morris Newman

Robert C. Thompson

Henryk Minc

Roy B. Leipnik

Marvin Marcus

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Research under the following general headings was concluded ~~during the period~~ 10/1/78 - 09/30/79 under Contract F49620-78-C-0030: (1) Eigenvalue inequalities (Marcus, Thompson); (2) Numerical range calculations (Marcus); (3) Exact solutions of integral systems of linear equations (Newman); (4) Experiments on the ILLIAC IV, Sunnyvale, CA (Newman); (5) The Campbell Baker Hausdorff formula (Thompson); (6) Rearrangement inequalities and determinant bounds (Minc); (7) Non-linear control in 3 x 3 linear systems (Leipnik).

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PUBLICATIONS

completed under AFOSR Contract No. AF F4962078-C-0030

October 1, 1978 - September 30, 1979

MARVIN MARCUS

1. An inequality for non-decomposable elements in the Grassmann algebra, Houston Journal of Math. 4(1978), 417-422.

2. Decomposable symmetrized tensors, (with J. Chollet), Linear and Multilinear Algebra 6(1978), 317-326.

3. Decomposable symmetrized tensors and an extended LR decomposition theorem, Linear and Multilinear Algebra 6(1978), 327-330.

4. Linear operators preserving the decomposable numerical range, (with I. Filippenko), Linear and Multilinear Algebra 7(1979), 27-36.

5. The numerical range of certain 0,1-matrices, (with B.N. Shure), Linear and Multilinear Algebra 7(1979), 111-120.

6. Some combinatorial aspects of the numerical range, Annals of the New York Academy of Sciences 319(1979), 368-376.

7. Variations on the Cauchy-Schwarz inequality, Linear Algebra and Appl. 27(1979), 81-91.

8. Inequalities connecting eigenvalues and non-principal subdeterminants, Proceedings of the Second International Conference on General Inequalities at Oberwolfach, (with I. Filippenko), to appear.

9. Linear groups defined by decomposable tensor equalities, (with J. Chollet), to appear.

10. A subdeterminant inequality for normal matrices, (with K. Moore), Linear Algebra and Appl., to appear.

11. A determinant formulation of the Cauchy-Schwarz inequality, (with K. Moore), to appear.

12. Some variations on the numerical range, (with Bo-Ying Wang), to appear.

MARVIN MARCUS

M. Marcus has published a total of 165 research papers and 16 books and research monographs since the beginning of his research career in 1954. His initial work was concerned with the behavior of solutions to systems of ordinary differential equations and appeared in the Princeton Annals of Mathematics Studies, Contributions to Nonlinear Oscillations between 1955 and 1960. Problems in stability theory for such systems can frequently be analyzed by studying the linear approximation. This, in turn, leads to classical problems in eigenvalue localization. Most of Marcus' work starting in the late 1950's has been in classical algebraic stability theory and its supporting and derivative disciplines: multilinear algebra; convexity theory; theory of inequalities; group representation theory; numerical algebra.

During the period of the contract (October 1, 1978-September 30, 1979), Marcus' research has been concerned with the following general problem which in turn separates into two parts. The beginnings of this work was presented in an invited address at Gatlinburg VII, Conference on Numerical Linear Algebra, December 11-17, 1977. Further developments were reported on in an invited address at the New York Academy of Sciences, Second International Conference on Combinatorial Mathematics, New York, April 4-7, 1978. The most recent work completed in the 1978-79 period was recently presented at an invited address at the California Institute of Technology. Marcus has been invited to an NSF sponsored Conference on Linear Algebra at Auburn to present a survey lecture on this research and later developments (March 1980).

GENERAL PROBLEM. *Let V be an n-dimensional unitary space with inner product $(\cdot,\cdot)$ and let $T:V \to V$ be a linear operator. Given a function $f:C^k \to C$, investigate $W_f(T)$, the set of points in the complex plane, C, of the form*

$$f((Tx_1,x_1),\ldots,(Tx_k,x_k)) .$$

The vectors $x_1, \ldots, x_k$ usually run over prescribed sets of k orthonormal vectors in V; however, sometimes the hypotheses stipulate the value of the inner product (x_i,x_j) to be something other than δ_{ij}, $i,j = 1, \ldots, k$.

If $k = 1$ and $f(z) = z$ then $W(T)$, the classical Toeplitz-Hausdorff numerical range of T, is a specific instance of $W_f(T)$. Thus $W_f(T)$ will be called the f-*numerical range of* T.

This General Problem separates into the following specific types of investigations:

I. *The relationship between the geometry of $W_f(T)$ and the eigenvalues, singular values, elementary divisors, etc. of T.*

II. *Numerical computations of $W_f(T)$ and related sets.*

Results related to I

In a recent query A. Abian posed the following interesting question. Let V be an n-dimensional inner product space, and let A, B, P, Q be linear on V. What relations must exist among these operators so that the inequality

$$(Av,u)(Bu,v) \leq (Pu,u)(Qv,v) \tag{1}$$

holds for all u and V ?

Marcus solved this problem completely in [7].

THEOREM. *Let* V *be a unitary space,* $\dim V \geq 3$, *and assume that* A, B, P, Q *are nonsingular. Then* (1) *holds for all* u *and* v *if and only if*

(i) $P = \alpha H$, $Q = \beta K$, $\alpha\beta = \epsilon = \pm 1$, H *and* K *are definite hermitian and*

(ii) $A^* = \lambda B$, λ *real, so that* (1) *reads*

$$\lambda |(Bu,v)|^2 \leq \epsilon(Hu,u)(Kv,v) , \tag{2}$$

and

(iii) *if* $\epsilon = 1$, $\lambda > 0$, *then* H *and* K *have the same sign* (*i.e.*, *both positive definite or both negative definite*) *and*

$$\lambda_{max}(P^{-1}AQ^{-1}B) \leq 1 ; \tag{3}$$

or

(iv) *if* $\epsilon = 1$, $\lambda < 0$, *then* H *and* K *have the same sign*; *or*

(v) *if* $\epsilon = -1$, $\lambda > 0$, *then* H *and* K *have opposite signs and*

$$\lambda_{max}(P^{-1}AQ^{-1}B) \leq 1 ; \tag{4}$$

or

(vi) *if* $\epsilon = -1$, $\lambda < 0$, *then* H *and* K *have opposite signs*.

This result was proved using properties of linear maps on the tensor space over V. The same techniques led to results of the following type:

Let $A_1, \ldots, A_k$ be linear maps on V, $\dim V = n$, and define the set

$$W(A_1,\ldots,A_k) = \left\{ \prod_{i=1}^{k} (A_i x_i, x_i) \,|\, x_1,\ldots,x_k \ \text{o.n.} \right\} .$$

THEOREM. *If* $A_1, \ldots, A_k$ *are linear maps on a unitary space* V, $k \leq n = \dim V$, *then*

$$W(A_1,\ldots,A_k) = \{0\}$$

iff some A_i *is* 0 [12].

Using properties of the Grassmann algebra the following results were obtained in [10].

Let A be an $n \times n$ normal matrix over $\mathbb{C}$, and $Q_{m,n}$ be the set of strictly increasing integer sequences of length m chosen from $1, \ldots, n$. For $\alpha, \beta \in Q_{m,n}$ denote by $A[\alpha|\beta]$ the submatrix obtained from A by using rows numbered α and columns numbered β. For $k \in \{0,1,\ldots,m\}$ write $|\alpha \cap \beta| = k$ if there exists a rearrangement of $1, \ldots, m$ say $i_1, \ldots, i_k, i_{k+1}, \ldots, i_m$ such that $\alpha(i_j) = \beta(i_j)$, $i = 1, \ldots, k$, and $\{\alpha(i_{k+1}),\ldots,\alpha(i_m)\} \cap \{\beta(i_{k+1}),\ldots,\beta(i_m)\} = \emptyset$. A new bound for $|\det A[\alpha|\beta]|$ is obtained in terms of the eigenvalues of A when $2m = n$, and $|\alpha \cap \beta| = 0$.

THEOREM. *Let* $m \geq 2$, $n = 2m$, *and let* A *be an* $n \times n$ *normal matrix over* $\mathbb{C}$ *with eigenvalues* $\lambda_1, \ldots, \lambda_n$. *Denote by* λ_ω *the product* $\lambda_{\omega(1)} \cdots \lambda_{\omega(m)}$, *for* $\omega \in Q_{m,n}$. *Then*

$$|\det A[\omega|\omega']| \leq \begin{cases} \dfrac{1}{4} \displaystyle\sum_{\omega \in Q} |\lambda_\omega + \lambda_{\omega'}| & \text{if } m = 2 \\[2ex] \dfrac{1}{2(m+1)} \displaystyle\sum_{\omega \in Q} |\lambda_\omega + (-1)^m \lambda_{\omega'}| & \text{if } m > 2 . \end{cases}$$

The set $Q \subset Q_{m,n}$ is any subset with $|Q| = \frac{1}{2}\binom{n}{m}$.
For example, if $m = 2$, $n = 4$ take $Q = \{(12),(13),(14)\}$, $Q' = \{(34),(24),(23)\}$ and the result states that $|\det A[12|34]| \leq \frac{1}{4}(|\lambda_1\lambda_2 + \lambda_3\lambda_4| + |\lambda_1\lambda_3 + \lambda_2\lambda_4| + |\lambda_1\lambda_4 + \lambda_2\lambda_3|)$.

In [4] the following results are obtained. Let $E_m(|\lambda|) = E_m(|\lambda_1|,\ldots,|\lambda_n|)$ denote the mth elementary symmetric polynomial in $|\lambda_1|, \ldots, |\lambda_n|$, i.e.,

$$E_m(|\lambda|) = \sum_{\omega \in Q_{m,n}} \prod_{i=1}^{n} |\lambda_{\omega(i)}| .$$

THEOREM. *Let* A *be a normal matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$, $n \geq 4$. *Let* B *be an* m-*square submatrix of* A *having precisely* k *main diagonal entries lying on the main diagonal of* A. *If* $k \leq m-2$ *then*

$$|\det B| \leq \begin{cases} \dfrac{E_m(|\lambda|)}{2(m-k+1)} & \text{if } k < m-2 \\[2ex] \dfrac{E_m(|\lambda|)}{4} & \text{if } k = m-2 . \end{cases}$$

Some corollaries that are consequences of this result are directly applicable to eigenvalue localization problems.

Recall that the *spread* of A is the number

$$s(A) = \max_{i,j} |\lambda_i - \lambda_j| .$$

Corollary 1. *If* $2 < m < n$ *and the* m-*square submatrix* B *of* A *has no main diagonal elements lying on the main diagonal of* A *then*

$$s(A) \geq \sqrt{3}\left(2(m+1)\binom{n}{m}^{-1}\right)^{1/m} |\det B|^{1/m} , \tag{5}$$

and if $m = 2$

$$s(A) \geq 2\sqrt{6}(n(n-1))^{-1/2} |\det B|^{1/2} .$$

In the second corollary A is an arbitrary n-square matrix, $(n \geq 2)$. Let d_m be the largest m-square subdeterminant of A (in absolute value) and let $\|A\|$ be the Hilbert norm of A, i.e., the largest singular value of A.

Corollary 2. <u>If</u> $2 < m \leq n$ <u>then</u>

$$\|A\| \geq \left(2(m+1)\binom{2n}{m}^{-1}\right)^{1/m} d_m^{1/m} ,$$

<u>and</u>

$$\|A\| \geq 2(n(n-1))^{-1/2} d_2^{1/2} .$$

<u>Results related to II</u>

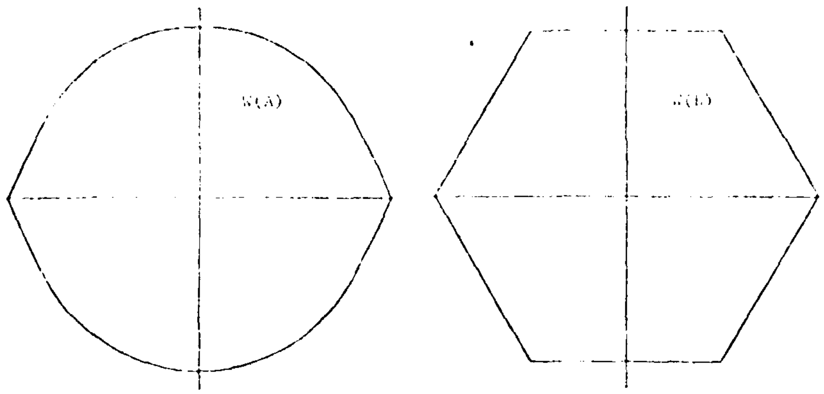
In [5] the following results were obtained:

THEOREM. *Let A be a 0,1-matrix with at most one 1 in each row and column. Then the numerical range of A is the convex hull of a polygon and a circular disk, both centered at the origin and contained in the unit disk.*

The proof uses a permutation similarity to reduce A to a direct sum of matrices whose numerical ranges can be determined. A computer program was developed which plots the boundary of the numerical range of an arbitrary complex matrix. For example, for $n = 10$ consider the matrix

$$A = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \end{bmatrix}$$

To exhibit the sensitivity of W(A) to changes in A let B be the matrix obtained from A by replacing the 0 in the (5,6) position by a 1. The following are plots of W(A) and W(B) produced by a program using standard FORTRAN subroutines and run on the AS/6 at the UCSB Computer Center.



An unresolved conjecture of Marcus [10] states that for a normal matrix with given eigenvalues the closer a subdeterminant is to being principal, the larger its absolute value can be. More precisely let A be an $n \times n$ matrix and consider the $m \times m$ subdeterminants of U^*AU lying in rows $1, \ldots, m$, columns $1, \ldots, k, m+1, \ldots, 2m-k$:

$$|\det U^*AU[1,\ldots,m|1,\ldots,k,m+1,\ldots,2m-k]| \ ;$$

U runs over all unitary matrices. The columns labeled $k = 0$, $k = 1$, $k = 2$, $k = 3$ and $k = 4$ in the following Table contain the maximum values obtained for these numbers. One hundred random unitary matrices were generated for each run: The eigenvalues of A (the only pertinent information) are listed in the left hand column. The 100 'random' unitary matrices were generated as follows:

1. $n(n+1)/2$ random complex numbers were generated to obtain an $n \times n$ skew-hermitian matrix S.
2. Then standard FORTRAN subroutines were used to compute the Cayley transform

$$U = (I_n - S)(I_n + S)^{-1} ,$$

a unitary matrix.

TABLE

A	n	m	k=0	k=1	k=2	k=3
1,1/2,1/3,1/4	4	2	.03667	.11762	.41957	
1,2,3,4	4	2	.88002	2.3705	8.4458	
10,100,0,.001	4	2	243.29	392.61	769.71	
1,0,0,1/4	4	2	.05748	.07299	.11471	
1,2,3,4,5	5	2	1.0314	3.6461	12.359	
1,0,1/2,10,5	5	2	7.6828	11.539	19.721	
1,2,3,4,5,6	6	2	1.6436	4.2506	14.266	
1,2,3,40,5,100	6	2	620.57	666.20	1098.9	
1,1/2,1/3,1/4,1/5,1/6	6	2	.03345	.09892	.38888	
1,1/2,1/3,1/4,1/5,1/6	6	3	.00216	.00621	.02610	.13324
1,2,3,40,5,100	6	3	1098.2	1386.3	1624.5	2959.7
1000,100,10,1,0,.01	6	3	67,324	105,170	221,070	650,600
30,20,10,0,.01,.001	6	3	434.84	675.60	1450.8	3892.9
1,1/2,1/3,1/4,1/5,1/6,1/7,1/8	8	2	.02012	.07197	.37369	
1,1/8,1/2,1/7,1/3,1/6,1/4,1/5	8	2	.04512	.08232	.29116	
1,2,3,4,5,6,7,8	8	2	1.9060	6.3162	19.145	
8,1,2,70,3,6,55,1/3	8	2	226.97	266.19	378.16	
1,1/2,1/3,1/4,1/5,1/6,1/7,1/8	8	3	.00203	.00503	.02261	.12924
1,1/8,1/2,1/7,1/3,1/6,1/4,1/5	8	3	.00220	.00856	.01779	.10749
80,10,70,60,50,30,20,40	8	3	3442.2	12,925	47,310	201,440
1,2,3,4,5,6,7,8	8	3	3.0235	4.2983	15.175	74.860
8,1,2,70,3,6,55,1/3	8	3	944.15	1769.7	3690.2	4351.8

TABLE

A	n	m	k=0	k=1	k=2	k=3	k=4
1,1/2,1/3,1/4,1/5,1/6,1/7,1/8	8	4	.00011	.00027	.00119	.00634	.03240
1,1/8,1/2,1/7,1/3,1/6,1/4,1/5	8	4	.00011	.00033	.00112	.00494	.01757
80,10,70,60,50,30,20,40	8	4	51,800	216,955	480,249	2,480,682	10,100,225
1,2,3,4,5,6,7,8	8	4	4.4268	5.7196	21.873	89.545	282.17
2,4,90,0,0,1/2,85,1/4	8	4	1592.4	2512.7	3027.8	8223.4	14,684
8,1,2,70,3,6,55,1/3	8	4	2162.9	5253.0	7693.2	9794.0	21,919

PUBLICATIONS

completed under AFOSR Contract No. AF F4962078-C-0030

October 1, 1978 - September 30, 1979

MORRIS NEWMAN

1. Combinatorial matrices with small determinants, Canad. J. Math. 30 (1978), 756-762.

2. Normalizers of modular groups, Math. Ann. 238(1978), 123-129.

3. Equivalence without determinantal divisors, Linear and Multilinear Algebra 7(1979), 107-109.

4. The use of integral operators in number theory (with C. Ryavec and B.N. Shure), J. Functional Analysis 32(1979), 123-130.

5. Roots of unity in finite fields, to appear.

6. Condition numbers of integral symmetric matrices, to appear.

7. On a homogeneous diophantine equation, to appear.

8. A matrix factorization theorem, to appear.

9. Simple groups of square order and an interesting sequence of primes, to appear in Acta Arith.

10. A note on cospectral graphs (with C.R. Johnson), to appear in J. Comb. Theory.

11. On a problem suggested by Olga Taussky-Todd, to appear in Illinois J. Math.

12. Positive definite matrices and Catalan Numbers (with F.T. Leighton), to appear in Proc. Amer. Math. Soc.

13. Matrices of finite period and some special linear equations, to appear in Linear and Multilinear Algebra.

14. Gersgorin revisited, to appear in Letters in Linear Algebra.

15. A surprising determinantal inequality for real matrices (with C.R. Johnson), to appear in Math. Ann.

16. A radical diophantine equation (with Feng Xuning).

17. Determinants of circulants of prime power order, to appear in Linear and Multilinear Algebra.

MORRIS NEWMAN

The principal object of Newman's research is to apply number-theoretic ideas to problems of numerical analysis and computation. In particular, applications of modular arithmetic are being made to such topics as (1) the exact solution of an integral system of linear equations, (2) the determination of the exact inverse of an integral matrix, (3) the determination of the rank and a basis for the null space of an integral matrix, (4) the determination of the eigenvalues of a rational triple diagonal matrix to any desired degree of accuracy, and (5) the computation of the permanent of a matrix. Programs to perform (1) and (2) which take full advantage of parallel computation have been prepared for ILLIAC IV. For example, the exact solution of any integral linear system in at most 40 unknowns, together with the determinant, can be found in less than 10 seconds. ILLIAC IV programs for (3), (4), and (5) are in process of preparation.

Past Accomplishments Under The Existing Contract

The primary objective of the research undertaken in the last period was to produce an ILLIAC IV program to find the exact inverse of an $n \times n$ nonsingular matrix A with integer entries. This has been achieved. The program requires just $n^2 + 2n$ cells in each processing element of ILLIAC IV for storage, and takes full advantage of the parallel processing capabilities of the machine. For example, the exact inverse of any 40×40 matrix of this type can be found in approximately 30 seconds. The program actually produces the determinant and adjugate matrix, and may be combined with another existing program to compute $A^{-1}B$, where B is any $n \times k$ integral matrix. The time to perform a similar calculation on a serial machine could well be as high as 3 hours.

Contacts with Air Force Laboratories

Newman has had invaluable help from the staff of the Institute for Advanced Computation at Sunnyvale, and has also discussed the project with Paul Nikolai, at Wright-Patterson Air Force Base.

PUBLICATIONS

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October 1, 1978 - September 30, 1979

ROBERT C. THOMPSON

1. Principal minors of complex symmetric and skew matrices, Linear Algebra and Appl., (Householder dedication issue) 28(1979), 249-255.
2. Interlacing inequalities for invariant factors, Linear Algebra and Appl. 24(1979), 1-31.
3. Singular values and diagonal elements of complex symmetric matrices, Linear Algebra and Appl. 24(1979), 65-106.
4. The case of equality in the matrix valued triangle inequality, Pacific J. of Math. 82(1979), 279-280.
5. Invariant factors under a rank one perturbation, Canadian J. Math., in press, 12 typed pages.
6. The congruence numerical range, Linear and Multilinear Algebra, in press, 15 typed pages.
7. Invariant factors of complementary minors of integral matrices, Houston Journal of Math., in press, 7 typed pages.
8. The Smith invariants of a matrix sum, Proceedings of the American Math. Society, in press, 6 typed pages.
9. The congruence numerical range, Linear and Multilinear Algebra, in press, 15 typed pages.
10. p-adic matrix valued inequalities, in preparation, 12 typed pages.
11. A matrix exponential formula, Linear and Multilinear Algebra, in preparation, 10 typed pages.
12. The Smith form, the inversion rule for 2 x 2 matrices, and the uniqueness of the invariant factors for finitely generated modules, in preparation, 5 typed pages.
13. The Jacobi-Gundelfinger-Frobenius-Iohvidov rule and the Hasse symbol, to be submitted in Letters in Linear Algebra, 6-8 typed pages.
14. A matrix block diagonalization in the presence of a semidefiniteness hypothesis, (with Barbara LiSanti), in preparation.

Diagonal Elements of Complex Symmetric Matrices

It was proposed to study the relations between the diagonal elements and singulars of complex symmetric matrices. (Recall that the singular values of a matrix A are the eigenvalues of $(AA^*)^{1/2}$, A^* the conjugate transpose of A.) This problem arose from elementary particle physics, where the matrix A in Fermi statistics is complex symmetric and unitary. A complete solution was obtained, the form of the solution being unusually striking and unexpected. Specifically, let the diagonal elements be $d_1, d_2, \ldots, d_n$, numbered such that $|d_1| \geq \cdots \geq |d_n|$, and let the singular values be $s_1 \geq \cdots \geq s_n$. (These are nonnegative numbers.) Then a complex symmetric matrix exists with diagonal elements $d_1, \ldots, d_n$ and singular values $s_1, \ldots, s_n$, if and only if

$$
\begin{aligned}
|d_1| &\leq s_1 , \\
|d_1| + |d_2| &\leq s_1 + s_2 , \\
&\;\;\cdots \\
|d_1| + \cdots + |d_n| &\leq s_1 + \cdots + s_n , \\
|d_1| + \cdots + |d_{n-1}| - |d_n| &\leq s_1 + \cdots + s_{n-1} - s_n , \\
|d_1| + \cdots + |d_{n-2}| - |d_{n-1}| - |d_n| &\leq s_1 + s_{n-2} - s_{n-1} + s_n , \\
&\;\;\vdots \\
-|d_1| - \cdots - |d_n| &\leq -s_1 + s_2 + s_3 + \cdots + s_n ,
\end{aligned}
$$

and one further singularity,

$$
\begin{aligned}
|d_1| + \cdots + |d_{n-3}| - |d_{n-2}| - |d_{n-1}| - |d_n| \\
\leq s_1 + \cdots + s_{n-2} - s_{n-1} - s_n .
\end{aligned}
$$

(The last condition has three minus terms on the left and two on the right.)

The important part about this result is, not so much its unexpected form, but the fact that connections with the structure theory of simple Lie algebras are suggested. It is plain that underlying these inequalities is a root system for some simple Lie algebra, and work is now under way to identify this connection.

Matrix Valued Inequalities and Equalities; an Exponential Formula

During the term of the grant, the proposer was led to the following conjecture.

Let A and B be matrices, real or complex, of the same size. Then one would like to have

$$e^A e^B = e^{A+B} .$$

But of course this rarely happens, unless the matrices commute. However, on the basis of his experience with matrix valued inequalities, (see below), the proposer conjectured that

$$e^A e^B = e^{UAU^* + VBV^*} ,$$

for suitable unitary matrices dependent on A and B. Considerable work has been carried out on this conjecture, with these results:

(a) the conjecture has been verified in many cases,

(b) it has been formally verified, meaning that infinite series for U and V have been found that work when used in conjunction with the infinite series for e^x.

All that remains is to prove that the infinite series giving U and V are convergent. They seem to converge in examples, but strenuous efforts to establish this in general have so far not succeeded.

This problem is basically Lie-theoretic, the central tool being the Campbell Baker Hausdorff formula from Lie theory.

The Matrix Valued Triangle Inequality

If A is a matrix, define its matrix absolute value as $|A| = (AA^*)^{1/2}$. Let B be another matrix. In analogy with scalars, one would like to have the triangle inequality

$$|A+B| \leq |A| + |B| ,$$

meaning that the right side minus the left side is positive semidefinite. This inequality is usually false. The proposer has shown that it becomes true if it is changed to

$$|A+B| \leq U|A|U^* + V|B|V^* \qquad (1)$$

where U, V are unitary matrices dependent on A and B. During the course of the grant, the proposer discovered that this inequality shares all the properties of the scalar triangle inequality:

> Equality holds in (1) if and only if A and B have polar factorizations with a common unitary factor.

Further work is continuing on matrix valued inequalities.

Other Work

A variety of related problems involving eigenvalues, singular values, matrix inequalities, have been examined and continue to be examined. There are ongoing efforts to tie up these investigations with Lie theory, with some success to date, but probably with more success in the future.

PUBLICATIONS

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HENRYK MINC

1. On a conjecture of R. F. Scott (1881), Linear Algebra and Appl. 28 (1979), 141-153.
2. An asymptotic solution of the multidimensional dimer problem, Linear and Multilinear Algebra, to appear.
3. Bounds for permanents and determinants, Linear and Multilinear Algebra, to appear.
4. Rearrangement inequalities, to appear.

HENRYK MINC

A substantial part of Minc's research concerns permanents and their applications. In paper [1] an old, outstanding conjecture of R.F. Scott is proved. This is a formula for the permanent of a special type of Cauchy matrices. It is proved by evaluating the eigenvalues of certain circulants and their determinants. These results are of independent interest. In [2] a new lower bound for the dimer problem is obtained. Although the bound, together with an upper bound previously obtained by Minc, leads to an asymptotic solution of the dimer problem, the all-important three dimensional case remains unsolved. In order to improve the known bounds for the three-dimensional problem, a large number of permanents of (0,1)-circulants were computed and tabulated, but conclusive results have not been obtained yet.

Schinzel obtained the following remarkable inequality for the determinant of a real matrix $A = (a_{ij})$:

$$|\det(A)| \leq \prod_{p=1}^{n} \max\left(\sum_{\substack{q=1 \\ a_{pq} \geq 0}}^{n} a_{pq} , \sum_{\substack{q=1 \\ a_{pq} \leq 0}}^{n} |a_{pq}| \right) .$$

Schinzel's formula was substantially improved by Johnson and Newman. In [3] their formula was used to obtain bounds for the absolute value of the determinants of complex matrices. Johnson and Newman also applied their method to permanents of real matrices and showed that if A is an $n \times n$ matrix such that the sum of the positive entries and the absolute value of the negative entries m each row do not exceed 1, then

$$|\text{per}(A)| \leq 2^{n/2} . \tag{*}$$

In [3] a rearrangement inequality is obtained, and used to strengthen the bound in (*). This rearrangement inequality is generalized in [4] where eight other new rearrangement inequalities are obtained. For example, it is shown that if $(a_{1j},\ldots,a_{m_j,j})$ are nonnegative m_j-tuples, $j = 1, \ldots, n$, $m_1 \geq \cdots \geq m_n \geq 1$, then

$$\prod_{j=1}^{n} \sum_{i=1}^{m_j} a_{ij} \geq \prod_{j=1}^{n} \sum_{i=1}^{m_j} \alpha_{ij}^{*},$$

where the α_{ij}^{*} are the numbers a_{ij} arranged in nondecreasing order, that is

$$\alpha_{11}^{*} \geq \alpha_{21}^{*} \geq \cdots \geq \alpha_{m_1,1}^{*} \geq \alpha_{12}^{*} \geq \alpha_{22}^{*} \geq \cdots \geq \alpha_{m_2,2}^{*} \geq \cdots \geq \alpha_{1n}^{*} \geq \cdots \geq \alpha_{m_n,n}^{*}.$$

This result and the other inequalities in [4] generalize and extend rearrangement inequalities obtained by Minc in 1971.

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ROY B. LEIPNIK

1. Book Review: Mathematics for Operations Research, W.H. Marlow, Wiley-Interscience, 1978, Linear and Multilinear Algebra 6(1978), 331-338.

2. Book Review: On Systems Analysis: An Essay Concerning the Limitations of Some Mathematical Methods in the Social, Political and Biological Sciences, David Berlinski, MIT Press, 1976, Linear and Multilinear Algebra 6(1978), 333-335.

3. Book Review: Applied Linear Algebra, Second Edition, Ben Noble and James Daniel, Prentice Hall, Linear and Multilinear Algebra 6(1978), 335-337.

4. Inversion of order in non-absolutely convergent multiple integrals, to appear.

5. Solution of the Kac-Siegert equations, to appear.

6. Application of the Kac-Siegert equations to distributions of spectral estimates, to appear.

7. Sampling versus quantizing in stochastic processes, to appear.

8. Schwarz distribution theory for linear difference-differential equations, to appear.

ROY B. LEIPNIK

The scalar discrete optimal gain solution discussed in the 1977-78 annual report was extended to 2×2 matrices with diagonal objective functions. Linear control of non-linear systems was accomplished for the case of 2×2 quadratic systems of Hille (prey-predator) type - this has ecological applications. Non-linear control of 3×3 linear systems was accomplished for the case of conflict between unsymmetrical adversaries (one aggressive, one defensive) and simulated for symmetrical adversaries (both aggressive). The linear control of space-craft attitude (non-linear equations) was illuminated by numerous analog computer studies but a satisfactory theory remains elusive.

The control of saturating lasers was studied by an integral equation method of T. Triffet (of Arizona,Tucson) and H.S. Green (of Adelaide, Australia). The results can be applied to genetic problems in which the reproduction ratios are variable, decreasing (but not to zero) as population densities increase. The details involve the spectra of rather complicated differential operators-in general, it can be said that the saturation is not exponential, but can be approximated by a combination of several exponentials; the decay coefficients are expressible in terms of physically (or genetically) significant quantities.

Some attention was given to obtaining the characteristic function of the lognormal distribution, and to finding the distribution of the sum of independent, but not identical, lognormal random variables. This work has numerous applications in radar detection and component reliability. The results are expressible in terms of series of Hermite functions with gamma (or incomplete gamma) function coefficients.